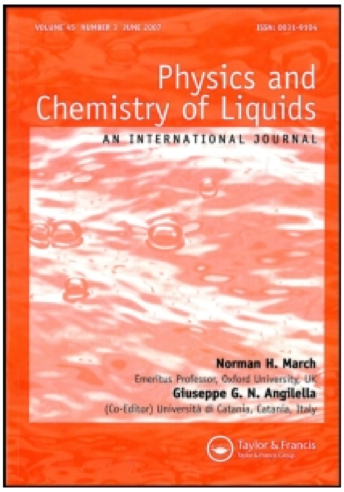

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Anyon liquid fractional statistics distribution function with applications to phonons in a low-dimensional vibrating lattice and to two-dimensional magnets

G. G. N. Angilella[a]; N. H. March[bc]; R. Pucci[a]
[a] Dipartimento di Fisica e Astronomia, Università di Catania, and Lab. MATIS-INFM, and CNISM, Sez. Catania, and INFN, Sez. Catania, I-95123 Catania, Italy [b] Oxford University, Oxford, UK [c] Department of Physics, University of Antwerp, Groenenborgerlaan 171, B-2020 Antwerp, Belgium

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Letter

Anyon liquid fractional statistics distribution function with applications to phonons in a low-dimensional vibrating lattice and to two-dimensional magnets

G. G. N. ANGILELLA*†, N. H. MARCH‡§ and R. PUCCI†

†Dipartimento di Fisica e Astronomia, Università di Catania,
and Lab. MATIS-INFM, and CNISM, Sez. Catania, and INFN, Sez. Catania,
Via S. Sofia, 64, I-95123 Catania, Italy
‡Oxford University, Oxford, UK
§Department of Physics, University of Antwerp,
Groenenborgerlaan 171, B-2020 Antwerp, Belgium

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Anyons have been used earlier to interpret the oscillatory orbital magnetization in the Laughlin two-dimensional (2D) electron liquid, which is the seat of the fractional quantum Hall effect. Here, we examine first the anyon fractional statistics distribution function in some detail. Then, applications are treated to other 2D systems, namely (a) a vibrating lattice such as a graphene layer in crystalline graphite, and (b) 2D insulating ferromagnets.

Keywords: Anyon fluids; Quantum statistics; Phonons; Magnetic excitations

1. Introduction

That anyonic fractional statistics become relevant in two-dimensional (2D) systems was demonstrated by Leinaas and Myrheim [1] (see also ref. [2]). An early application was then made by Lea *et al*. [3,4] to discuss the de Haas–van Alphen-like oscillatory orbital magnetism in the Laughlin electron liquid which is the seat of the fractional quantum Hall effect [5]. In parallel, a study was also made of the anyon fractional distribution function $f_\alpha(\epsilon)$, where ϵ denotes the particle energy [6–8]. These early proposals, which successfully interpolated between Bose–Einstein (BE) and Fermi–Dirac (FD) limits using a fractional statistics parameter α in the range between 0 and 1, BE corresponding to the end-point $\alpha = 0$ and FD to the other extreme $\alpha = 1$,

*Corresponding author. E-mail: giuseppe.angilella@ct.infn.it

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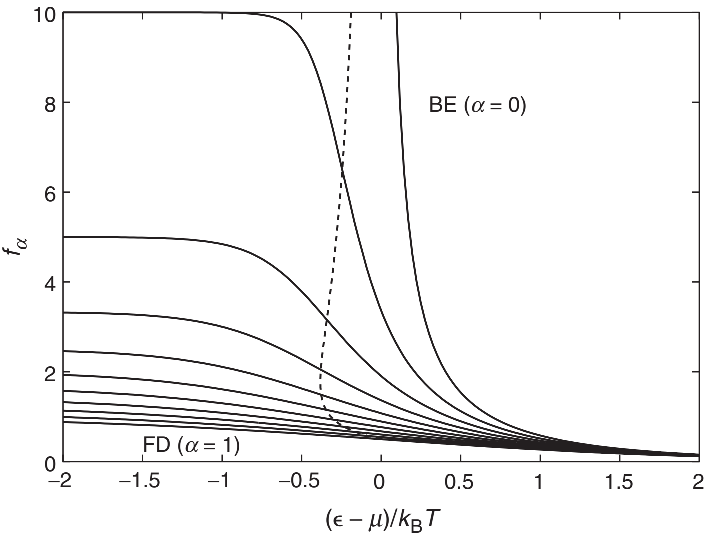

Figure 1. Distribution functions for anyon statistics, equation (1), at finite temperature $T \neq 0$, for equally spaced values of statistics parameter α ranging between $\alpha = 0$ (BE) and $\alpha = 1$ (FD). The dashed line connects the inflection points of the distribution functions in the quasifermionic cases ($0 < \alpha \leq 1$). Such points approximate the Fermi level.

were completed by the work of Wu [9]. Wu's distribution function for anyon statistics can then be presented as

$$f_\alpha(\epsilon) = \frac{1}{w[e^{(\epsilon-\mu)/k_B T}] + \alpha}, \tag{1}$$

where T is temperature, μ denotes the chemical potential, and the case $\alpha = \frac{1}{2}$ refers to 'semions'. In equation (1), the 'generalized exponential' $w(\zeta)$ obeys the functional equation [9]

$$w^\alpha(\zeta)[1 + w(\zeta)]^{1-\alpha} = \zeta \equiv e^{(\epsilon-\mu)/k_B T}, \tag{2}$$

and has been studied analytically by Joyce *et al.* in ref. [10]. Figure 1 shows plots of $f_\alpha(\epsilon)$ for $0 \leq \alpha \leq 1$. One recovers a quasifermionic behaviour for all values of α such that $0 < \alpha \leq 1$, with f_α characterized by an inflection point implicitly defined by the condition

$$w = -1 + \alpha + \sqrt{1 - \alpha + \alpha^2}. \tag{3}$$

Such an inflection point approximates the Fermi level as defined in ref. [9].

We turn now to apply the above distribution function results to 2D assemblies.

2. Specific heat contribution for a 2D vibrating model lattice: relevance to low-temperature specific heat of a graphene layer

Phonon properties are altered crucially at low temperatures as one goes from a truly three-dimensional material like diamond, with a lattice specific heat $C_V \propto T^3$ at low T, to a layered crystal like graphite, with only weak van der Waals interactions between graphene layers [11,12].

The vibrational properties of a discrete square lattice, including both nearest and next-nearest neighbours interactions, have been studied analytically by Montroll [13]. Montroll considers a discrete square lattice of equal masses M connected by harmonic 'springs' with Hookean constants κ' and κ'', say, between nearest and next-nearest neighbours, respectively. Born-von Kármán periodic boundary conditions are applied to such a lattice. Montroll then restricts his analysis to cases where $\tau = [1 + (\kappa'/2\kappa'')]^{-1} < \frac{1}{2}$, and particularly to the case $\tau = \frac{1}{3}$, where analytical results can be derived in closed form. As in Debye theory for the frequency spectrum of an elastic continuum, where an '*ad hoc*' cut-off frequency has to be advocated, Montroll naturally finds an upper bound $\nu_D = [(\kappa' + 2\kappa'')/\pi^2 M]^{1/2}$ to the spectrum of the allowed frequencies ν, as a consequence of the discrete nature of the lattice model he considers. The density of vibrational modes $g_M(\nu)$ can then be expressed as

$$g_M(\nu) = \frac{1}{2}[g_+(\nu) + g_-(\nu)], \tag{4}$$

where, for $\tau = \frac{1}{3}$, the densities of the two branches of allowed vibrational modes $g_\pm(\nu)$ assume the form

$$\begin{aligned}
\nu_D g_-(f\nu_D) &= \frac{24f}{\pi^2(2-3f^2)} K\left(\frac{3f^2}{2-3f^2}\right) \quad \text{if } 0 \le f^2 < \frac{1}{3}, \\
&= \frac{8}{\pi^2 f} K\left(\frac{2-3f^2}{3f^2}\right) \quad \text{if } \frac{1}{3} < f^2 \le \frac{2}{3}, \\
&= 0 \quad \text{if } f^2 > \frac{2}{3},
\end{aligned} \tag{5a}$$

$$\begin{aligned}
\nu_D g_+(f\nu_D) &= \frac{4f}{\pi^2\sqrt{1-\frac{4}{3}f^2}} K\left(f\sqrt{\frac{2-3f^2}{3-4f^2}}\right) \quad \text{if } 0 \le f^2 \le \frac{2}{3}, \\
&= \frac{8f}{\pi^2(1-f^2)} K\left(\frac{f\sqrt{f^2-(2/3)}}{1-f^2}\right) \quad \text{if } \frac{2}{3} < f^2 < \frac{3}{4}, \\
&= \frac{8}{\pi^2\sqrt{f^2-(2/3)}} K\left(\frac{1-f^2}{f\sqrt{f^2-(2/3)}}\right) \quad \text{if } \frac{3}{4} < f^2 \le 1, \\
&= 0 \quad \text{if } f^2 > 1,
\end{aligned} \tag{5b}$$

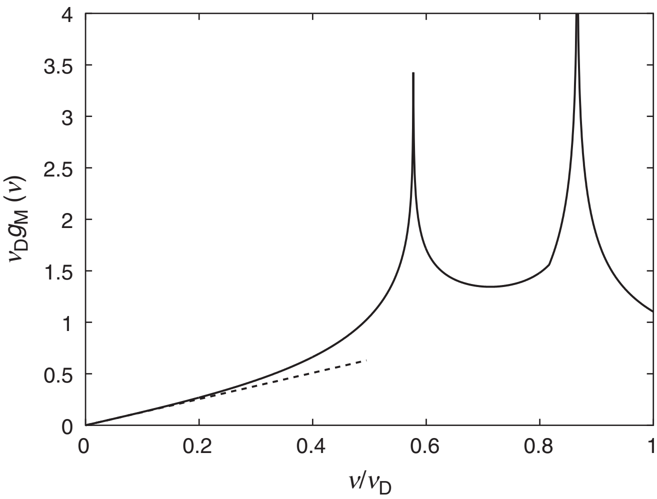

Figure 2. Density of vibrational states, equation (4), as in Montroll's model for a square lattice, with $\tau = \frac{1}{3}$ [13]. Note the two logarithmic singularities at $(\nu/\nu_D)^2 = \frac{1}{3}$ and $\frac{3}{4}$. The dashed line corresponds to the low-energy, long-wavelength, linear regime.

where $K(z)$ is the complete elliptic integral of the first kind [14]. Figure 2 shows equation (4) as a function of $f = \nu/\nu_D$. It should be noted that (i) $g_M(\nu)$ is normalized to unity, so that $\int_0^1 \nu_D g_M(f\nu_D)\mathrm{d}f = 1$; (ii) in the long-wavelength, low-energy regime the density of vibrational modes is linear in frequency, with $\nu_D g_M(f\nu_D) \sim 4f/\pi$; (iii) as a consequence of the general behaviour of the elliptic integrals, $g_M(\nu)$ is affected by (integrable) logarithmic singularities at $f^2 = \frac{1}{3}$ and $\frac{3}{4}$, both values occurring away from the range of frequencies where the linear regime applies ($f \lesssim 0.1$; cf figure 2). Such singularities have the same topological origin as van Hove singularities of electronic spectra in 2D lattices (see refs [15–17], and refs therein).

The internal energy per mode of an assembly of vibrational modes obeying fractional statistics, equation (1), and with frequency distribution given by equation (4), can then be expressed as

$$E_M(T) = \int_0^{\nu_D} h\nu g_M(\nu) f_\alpha(h\nu, T)\mathrm{d}\nu. \tag{6}$$

Since the asymptotic low-temperature behaviour of the relevant thermodynamic quantities (such as the internal energy, equation (6), and the specific heat at constant volume) is only determined by the long-wavelength, low-energy vibrational properties, we expect to recover a standard power-law T-dependence for such quantities in the limit $T \to 0$, with

$$(h\nu_D)^2 E_M(T) = a_M(k_B T)^3 + \cdots \tag{7}$$

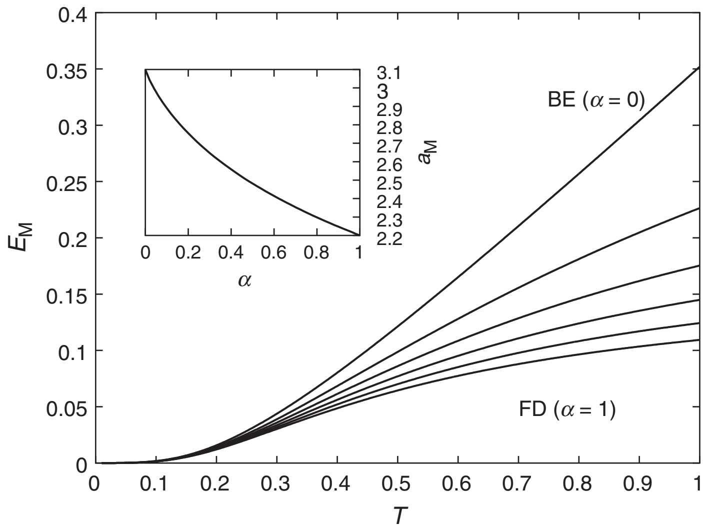

Figure 3. Internal energy per mode $E_M(T)$ *vs*. temperature T for Montroll's model of phonons in a 2D lattice, equation (6), for equally spaced values of the statistics parameter α between 0 (BE) and 1 (FD). Energy and temperature are in units of $h\nu_D$ and $h\nu_D/k_B$, respectively. Inset shows the numerical coefficient a_M in the low-temperature asymptotic expansion, equation (7), as a function of α.

[cf also equation (7) below]. Figure 3 shows our numerical result for $E_M(T)$, equation (6), as a function of temperature, and for the numerical prefactor a_M in its low-temperature expansion, equation (7). In particular, figure 3 confirms the internal energy T^3-law at low temperature, regardless of the value of the statistical parameter α, while the inset shows the actual α-dependence of the numerical prefactor a_M in equation (7). Such behaviours are recovered below in two further examples, namely vibrational modes of a continuum model and magnons, both in two dimensions.

3. Long wavelength properties: e.g. a single graphene layer

Since we expect sound waves to propagate at long wavelength in 2D lattices, the long wavelength limit of the phonon dispersion relation $\omega(\mathbf{q})$ in, say, a single hexagonal graphene layer must take the form

$$\omega(\mathbf{q}) = v_s^{\hat{\mathbf{q}}} q, \tag{8}$$

where $v_s^{\hat{\mathbf{q}}}$ is the velocity of sound. The corresponding density of vibrational states $g(\omega)$ is given by a d-dimensional Debye-like form

$$\begin{aligned} g_d(\omega) &= A_d \omega^{d-1}, \quad && \omega < \omega_D, && \text{(9a)} \\ &= 0, && \omega > \omega_D, && \text{(9b)} \end{aligned}$$

where ω_D is the Debye cut-off frequency. For $d=3$, we recover the usual Debye spectrum, while for $d=2$, our concern here, we have $g_2(\omega) = A_2\omega$. This linear dependence on ω is indeed recovered from Montroll's model discussed above in the limit $\omega \to 0$. From normalization arguments, the constant A_2 is determined, for a graphene layer, by the requirement

$$2N = \int_0^{\omega_\mathrm{D}} A_2 \omega \,\mathrm{d}\omega = \frac{1}{2} A_2 \omega_\mathrm{D}^2, \tag{10}$$

with N the number of C atoms in the graphene layer.

Hence the internal energy $E(T)$ is given by

$$\begin{aligned} E_2(T) &= \int_0^{\omega_\mathrm{D}} \hbar\omega g_2(\omega) f_\alpha(\hbar\omega, T) \mathrm{d}\omega \\ &= \int_0^{\omega_\mathrm{D}} \hbar\omega^2 A_2 f_\alpha(\hbar\omega, T) \mathrm{d}\omega. \end{aligned} \tag{11}$$

With $\alpha = 0$, $f_{\alpha=0}(\hbar\omega, T)$ becomes the BE distribution function with $\mu = 0$

$$f_\mathrm{BE}(\epsilon) = \frac{1}{\exp(\beta\epsilon) - 1}. \tag{12}$$

This will yield the internal energy $E_2(T)$ as $T \to 0$ as

$$E_2(T) = a_2 \frac{4Nk_\mathrm{B}^3}{\hbar^2 \omega_\mathrm{D}^2} T^3 + \cdots, \tag{13}$$

with $a_2(\alpha = 0) = 2\zeta(3) \approx 2.40411$, $\zeta(x)$ denoting Riemann's ζ-function.

Hence, from thermodynamics, the low-temperature specific heat C_V is given by

$$C_V = \frac{\mathrm{d}E_2(T)}{\mathrm{d}T} = 3a_2 \frac{4Nk_\mathrm{B}^3}{\hbar^2 \omega_\mathrm{D}^2} T^2 + \cdots, \tag{14}$$

a result that goes back at least to Krumhansl and Brooks [12] and also to Klein and Smith [18].

For the single graphene layer, as for a 2D system, we expect that anyon fractional statistics may obtain for the phonon quasiparticles, with a fractional statistics parameter α. Therefore, we have numerically performed the frequency integration in equation (11), and our results for $E_2(T)$ are shown in figure 4 [19]. In the low-temperature limit, we again recover the T^3 asymptotic dependence of E_2 on temperature, equation (13), but now with a numerical coefficient a_2 interpolating between the value $2\zeta(3)$ for BE statistics ($\alpha = 0$), and $\frac{3}{2}\zeta(3) \approx 1.80309$ for FD statistics ($\alpha = 1$; cf inset in figure 4). Therefore, the introduction of anyon statistics does not modify the exponents in the power laws of the low-temperature asymptotic behaviour of the relevant thermodynamical quantities, as could be expected from a dimensional analysis of equation (11) (noting that the statistical distribution f_α is dimensionless). Both the overall temperature dependence of the internal energy and the α-dependence of the

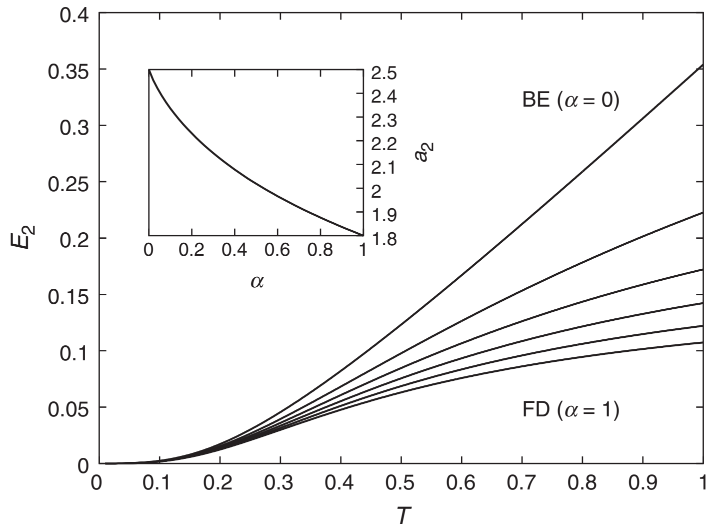

Figure 4. Internal energy $E_2(T)$ *vs.* temperature T for long-wavelength phonons in a 2D lattice, equation (11), for equally spaced values of the statistics parameter α between 0 (BE) and 1 (FD). Energy and temperature are in units of $4N\hbar\omega_D$ and $\hbar\omega_D/k_B$, respectively. Inset shows the numerical coefficient a_2 in the low-temperature asymptotic expansion, equation (13), as a function of α.

numerical prefactor a_2 in the low-T expansion equation (13) in figure 4 agree with their counterparts for anyonic vibrational frequencies of a discrete lattice, figure 3.

One should refer here to the very thorough discussion of both calculations and measurements of the phonon dispersion of graphite that has been given recently by Wirtz and Rubio [20]. A brief mention, in addition to this study of a graphene layer, should be made to phonons in MgB_2 [21], another layered material of considerable interest currently because of its superconductivity up to $T_c = 39.4\,\mathrm{K}$ [22].

4. Insulating 2D ferromagnet: low temperature spin wave excitations

Bloch's famous $T^{3/2}$-law [23] for the temperature dependence of the magnetization $M(T)$ of a 3D insulating ferromagnet, namely (see, e.g., ref. [24])

$$M(T) = M(0) - m_1 T^{3/2} + \cdots, \tag{15}$$

comes from the long-wavelength magnon dispersion relation

$$\omega(\mathbf{q}) = sq^2 + \cdots. \tag{16}$$

In less than three dimensions, it has been shown by Bloch [23] that such systems with reduced dimensionality should not exhibit spontaneous magnetization, even in cases where the exchange integral is positive. This is in agreement with the celebrated Mermin–Wagner theorem [25], which forbids the occurrence of truly off-diagonal

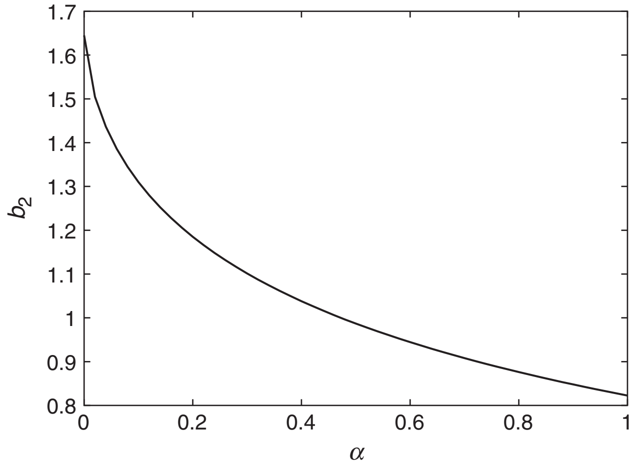

Figure 5. Numerical coefficient b_2 in the low-temperature dependence of the magnon internal energy, equation (18).

long-range order in 1D and 2D systems, even at very low temperatures, because of fluctuations. Therefore, Klein and collaborators [26,27] (see also ref. [28]) studied the temperature dependence of the magnetization of a thin ferromagnetic film. As the film thickness decreases, they found a crossover from the 3D power law, $M(0) - M(T) \propto T^{3/2}$, equation (15), to a power law with modified exponent, $M(0) - M(T) \propto T$, for film thicknesses below some 10 atomic layers.

Equation (16) leads to a constant 'density of states', ρ say, of such magnons at low frequencies and thus the internal energy $E(T)$ is given at low temperatures by

$$E(T) = \int_0^\infty \epsilon f_\alpha(\epsilon, T)\rho \, \mathrm{d}\epsilon \tag{17}$$

in 2D, where f_α is the anyon fractional statistics distribution function written above in equation (1), with its T dependence now noted explicitly. One finds

$$\frac{E(T)}{\rho} = b_2(k_\mathrm{B}T)^2, \tag{18}$$

with b_2 decreasing from the BE limit $b_2(\alpha = 0) = \zeta(2) = \frac{\pi^2}{6} \approx 1.64493$, to the FD limit $b_2(\alpha = 1) = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12} \approx 0.822467$. Figure 5 shows the numerical coefficient b_2 as a function of the statistics parameter α.

5. Summary

We have raised here the question as to whether anyon fractional statistics may need to be invoked for quasiparticles in 2D systems: especially phonons and magnons. To study whether experiment could tell whether such quasiparticles needed to have an associated fractional statistics parameter, we have investigated, in addition to a model 2D vibrating lattice solved analytically for the phonon spectrum by Montroll [13], the low temperature specific heat of a single graphene layer. In particular, we have explored numerically how $C_V \propto T^n$, with $n = 2.00 \pm 0.05$ found experimentally, [11] could embrace the introduction of a fractional statistics parameter α. Correspondingly, we find a low-temperature asymptotic expansion $E_2 \propto T^{n+1}$, invariably with $n = 2$, regardless of the value of the statistics parameter α, but with a slowly decreasing numerical coefficient as α increases from 0 (BE) to 1 (FD).

A briefer discussion of magnons has also been included. Again, one finds the same power-law temperature dependence of the main thermodynamic quantities (internal energy and specific heat) in the low-T regime, with a numerical coefficient weakly depending on the statistics parameter α.

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